22485 Medical Imaging systems

Lecture: Two-dimensional signal analysis

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Topic of today: Two-dimensional signal analysis

1. 2D Fourier transforms
   (a) Relation to 1D Fourier transforms
   (b) Examples and filtration
2. Hankel transform
3. Sampling of images
   (a) Spatial sampling frequencies
   (b) Discrete Fourier transforms
   (c) Circular convolution
4. Discussion of exercise 4 about spectral velocity estimation
5. Questions for assignments

Reading material: L. Prince & J. M. Links: Medical imaging signals and systems, Pearson Prentice Hall Bioengineering, 2006 or 2015, Chapter 2 and 3.

One dimensional Fourier transforms

Fourier transforms:

$$G(f) = \int_{-\infty}^{+\infty} g(t) e^{-j2\pi ft} dt$$

Inverse Fourier transforms:

$$g(t) = \int_{-\infty}^{+\infty} G(f) e^{j2\pi ft} df$$

If period is T then the frequency is $f_0 = 1/T$ in Hz or 1/s = s^{-1}

Two dimensional Fourier transforms

Fourier transforms:

$$F(u,v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

Inverse Fourier transforms:

$$f(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

If duration is X then the frequency is $f_x = 1/X$ in 1/m = m^{-1}

Properties of two dimensional Fourier transforms

Complex: $F(u, v) = R(u, v) + jI(u, v)$

Magnitude and phase: $F(u, v) = M(u, v)e^{-j\phi(u,v)}$

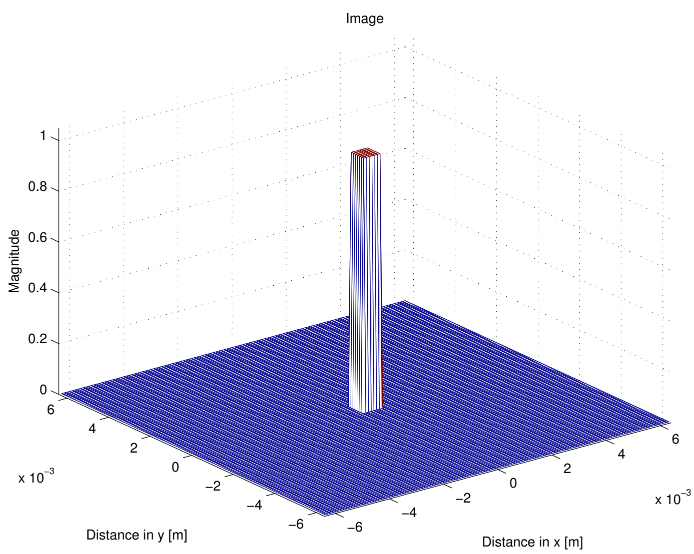
$M(u, v) = \sqrt{R(u, v)^2 + I(u, v)^2}$

$\phi(u, v) = \text{arctan}(I(u, v), R(u, v))$

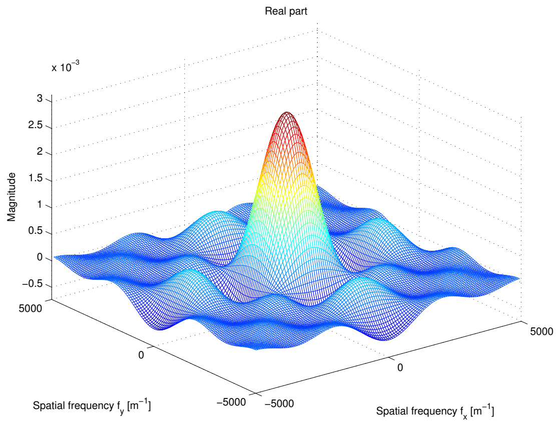
Calculation of Fourier transforms:

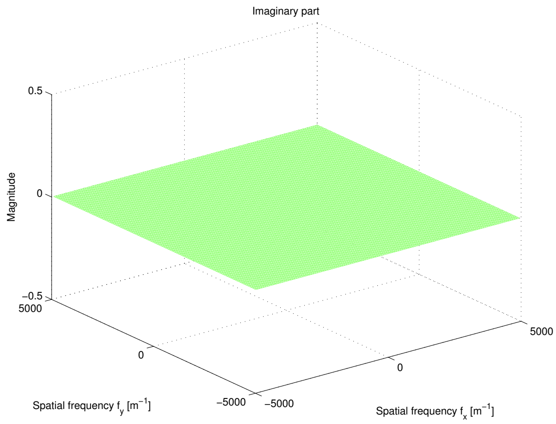
$$F(u, v) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(x, y)e^{-j2\pi ux}dx \right] e^{-j2\pi vy}dy$$

Two dimensional square

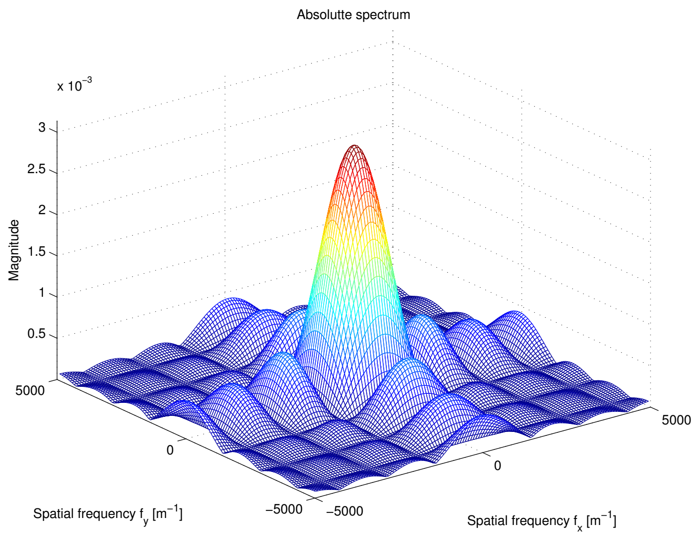


Two dimensional Fourier transforms - real and imaginary part





Two dimensional Fourier transforms - Magnitude



Properties of two dimensional Fourier transforms

Linearity: $af_1(x,y) + bf_2(x,y) \leftrightarrow aF_1(u,v) + bF_2(u,v)$

Shift: $f(x-a, y-b) \leftrightarrow F(u,v)e^{-j2\pi(ua+vb)}$

Convolution: $f_1(x,y) * f_2(x,y) \leftrightarrow F_1(u,v)F_2(u,v)$

$$f_1(x,y) * f_2(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_1(\alpha,\beta)f_2(x-\alpha, y-\beta)d\alpha d\beta$$

Multiplication: $f_1(x,y)f_2(x,y) \leftrightarrow F_1(u,v) * F_2(u,v)$

Separable image and Fourier transform

Image: $f(x,y) = f_x(x)f_y(y) \leftrightarrow F(u,v) = F_x(u)F_y(v)$

Example for square:

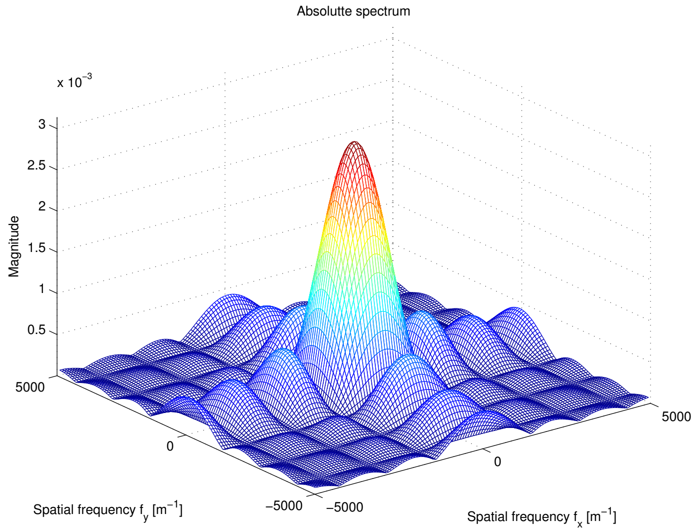
$f_x(x) = 1, |x| < k_x, \text{else } 0 \leftrightarrow F_x(u) = 2k_x\frac{\sin \pi u 2k_x}{\pi u 2k_x}$

$f_y(y) = 1, |y| < k_y, \text{else } 0 \leftrightarrow F_y(v) = 2k_y\frac{\sin \pi v 2k_y}{\pi v 2k_y}$

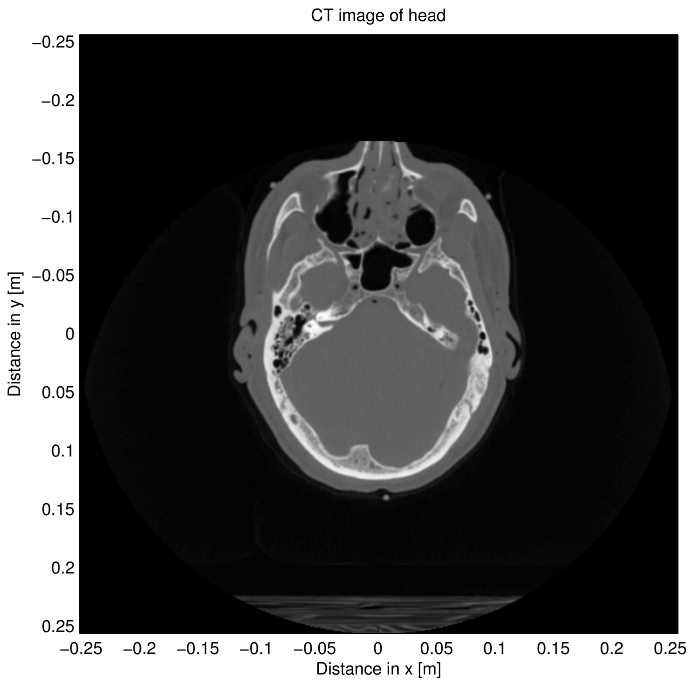
Spectrum for image:

$$f(x,y) = f_x(x)f_y(y) \leftrightarrow F(u,v) = F_x(u)F_y(v) = 2k_x\frac{\sin \pi u 2k_x}{\pi u 2k_x}2k_y\frac{\sin \pi v 2k_y}{\pi v 2k_y}$$

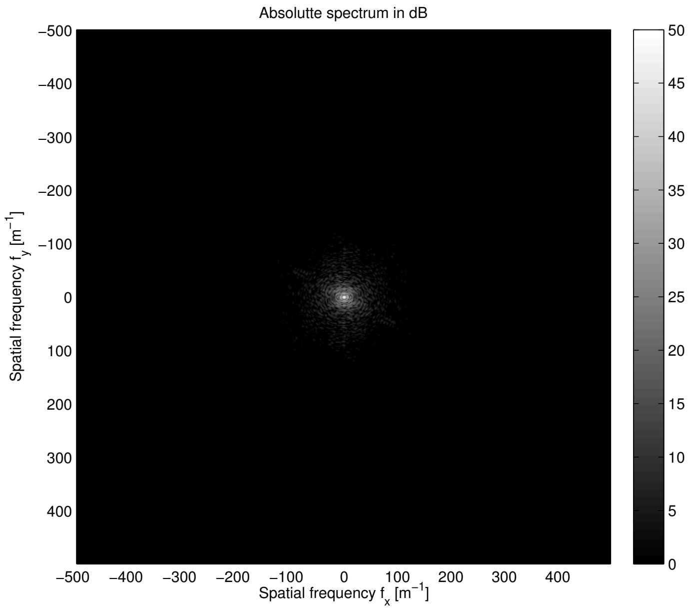
Two dimensional Fourier transforms - Magnitude



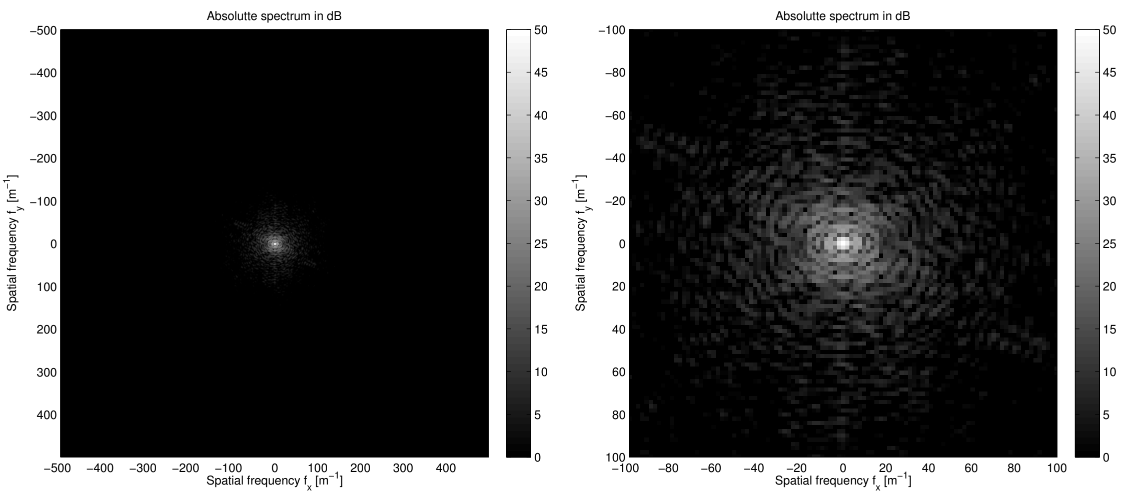
CT head image



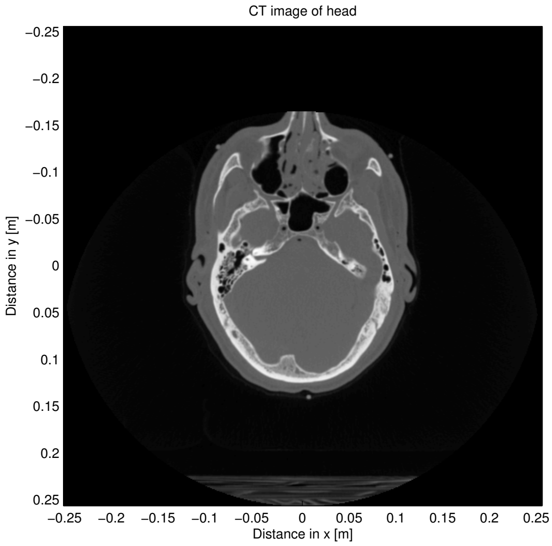
Two dimensional Fourier transforms - Magnitude

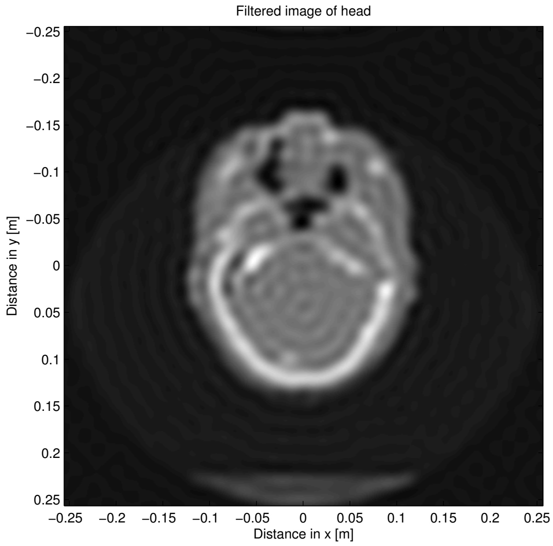


Two dimensional Fourier transforms - Magnitude zoomed

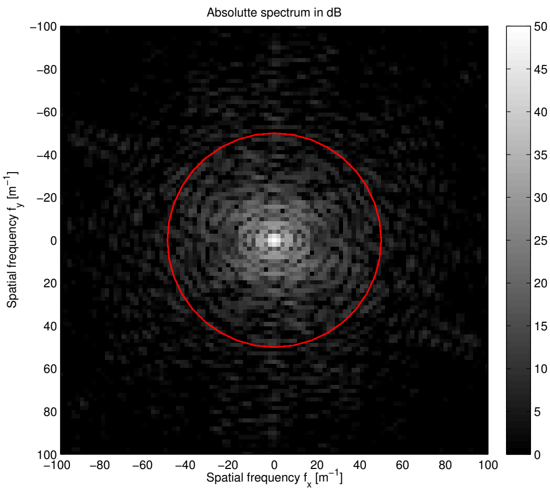


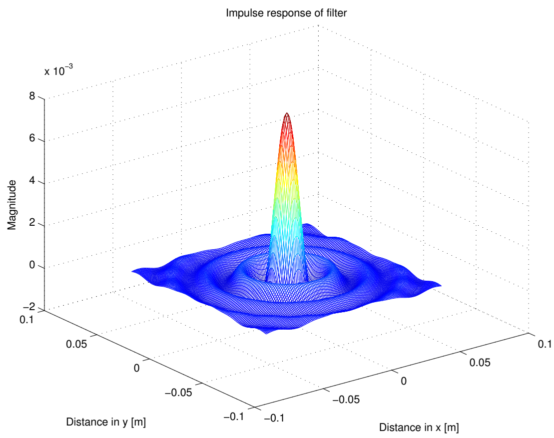
Low-pass filtration of head image





Low-pass filter for head image

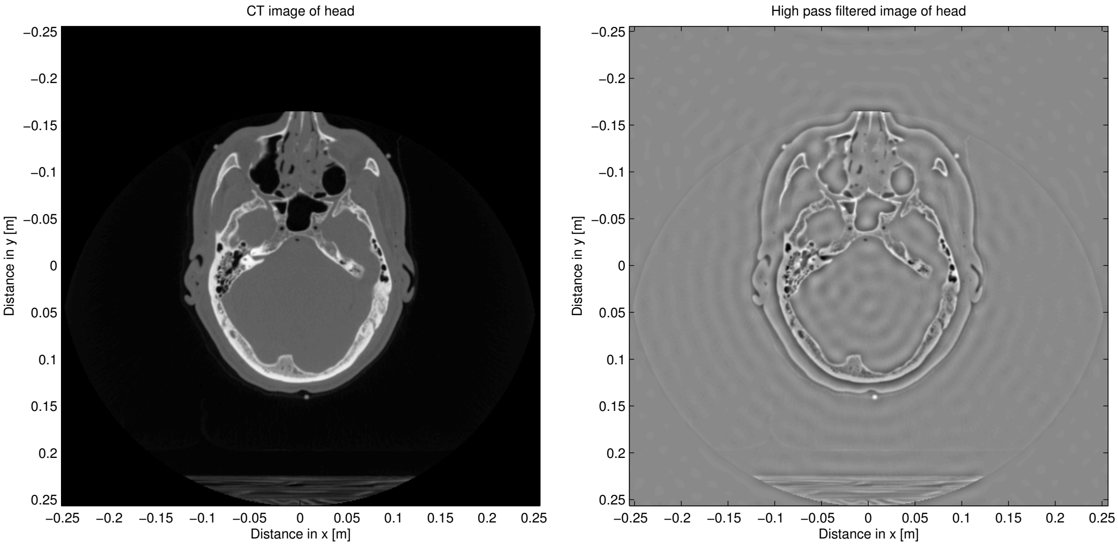




Spectrum in frequency domain

Impulse response

High-pass filtration of head image



Circular symmetric images - Hankel transform

Circular symmetric image: $f(x, y) = f(r, \phi) = f(r)$

Fourier transform circular symmetric: $F(u, v) = F(q, \Theta) = F(q)$

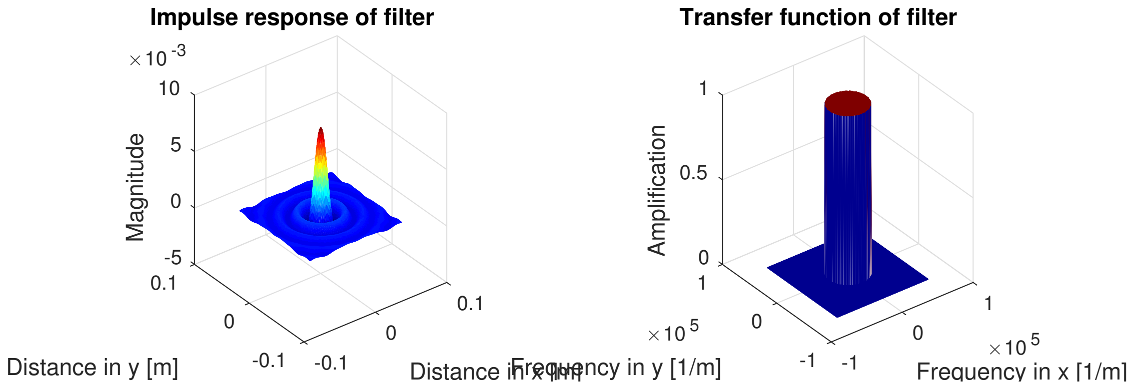
Given by the Hankel transform:

$$F(q) = 2\pi \int_0^{+\infty} r f(r) J_0(2\pi q r) dr$$

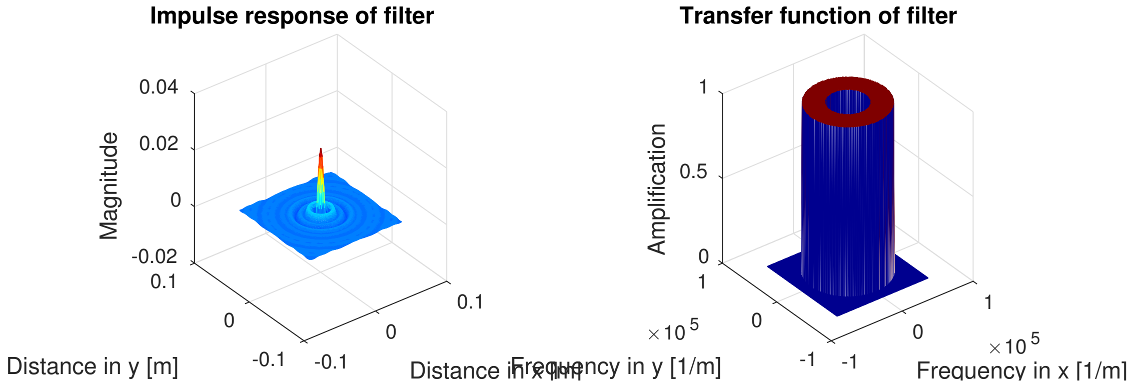
Bessel function of first kind:

$$J_0(r) = \frac{1}{\pi} \int_0^{\pi} \cos(r \sin \phi) d\phi$$

Circular symmetry - Hankel transform



Circular symmetry - Hankel transform



List of Hankel transforms

$$\exp(-\pi r^2) \leftrightarrow \exp(-\pi q^2)$$

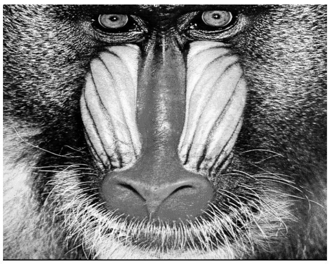
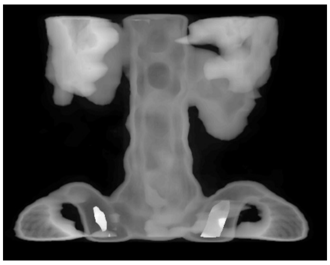
$$1 \leftrightarrow \frac{\delta(q)}{\pi q}$$

$$\delta(r-a) \leftrightarrow 2\pi J_0(2\pi q a)$$

$$\mathrm{sinc}(r) \leftrightarrow \frac{2\mathrm{rect}(q)}{\pi\sqrt{1-4q^2}}$$

Discussion after break

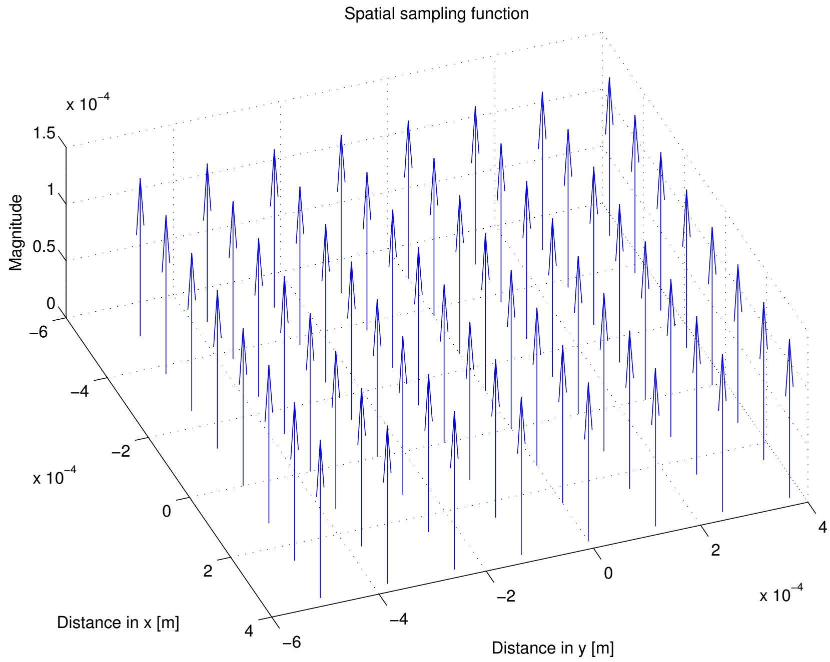
Consider what image you will see, if you combine the phase from one image with the amplitude from another image.

1. What is likely the amplitude spectrum of both images?
2. How would the phase be?
3. What is the combination?

for_10_2d_signals/matlab_demo/phase_demo.m

Sampling function



Sampling

2D image of δ-functions:

$$s(x,y) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \delta(x - j\Delta x, y - k\Delta y)$$

Spatial sampling frequencies: $f_{sx} = 1/\Delta x,\qquad f_{sy} = 1/\Delta y$

Sampled image:

$$f_s(x,y) = f(x,y)s(x,y) \leftrightarrow F(u,v) * S(u,v)$$

Fourier transform of sampling function:

$$s(x,y) \leftrightarrow S(u,v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(u - mf_{sx}, v - nf_{sy})$$

Spectrum of sampled image

Remember

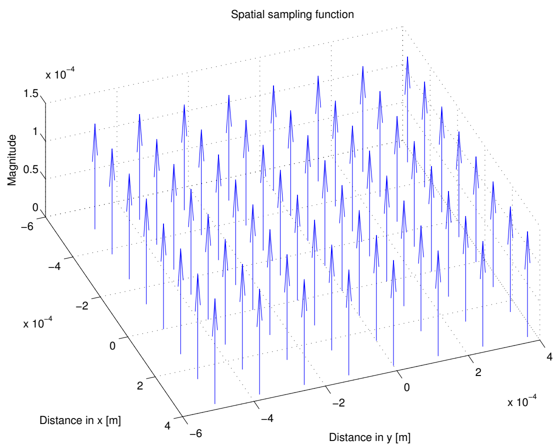
$$F(u,v) * \delta(u - mf_{sx}, v - nf_{sy}) = F(u - mf_{sx}, v - nf_{sy})$$

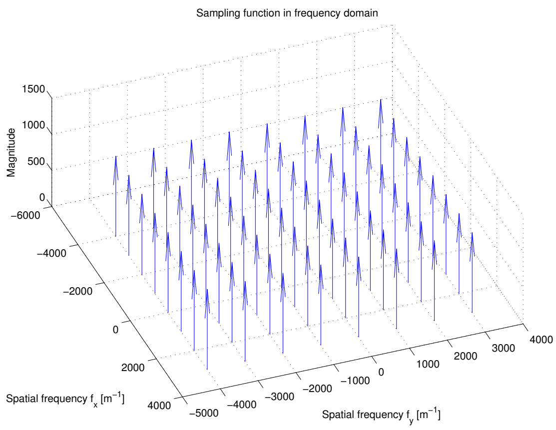
Spectrum

$$\begin{aligned} f_s(x,y) = f(x,y)s(x,y) &\leftrightarrow F(u,v) * S(u,v) \\ &= F(u,v) * \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(u - mf_{sx}, v - nf_{sy}) \\ F_s(u,v) &= \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} F(u - mf_{sx}, v - nf_{sy}) \end{aligned}$$

Spectrum repeats itself with a period of f_{sx} and f_{sy}

Sampling function

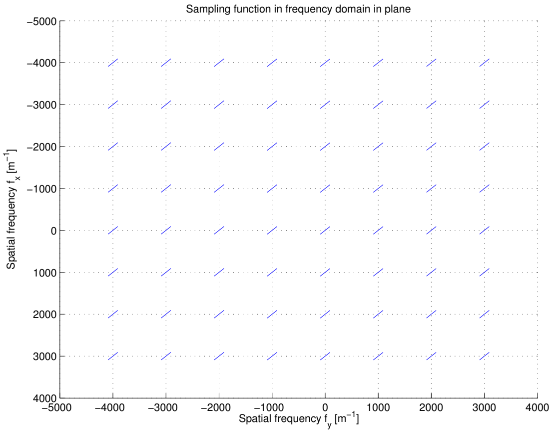


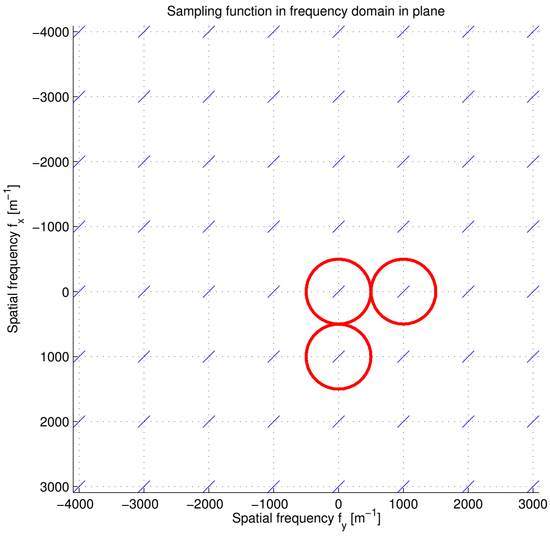


Spatial domain

Spectrum in frequency domain

Sampling function





Spatial Aliasing

Discrete two dimensional Fourier transforms

Discrete Fourier transforms:

$$F(u_d, v_d) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) e^{-j2\pi\left(\frac{mu_d}{M}+\frac{nv_d}{N}\right)}$$

Frequency variables are discrete: $u_d = 0..M-1$ and $v_d = 0..N-1$

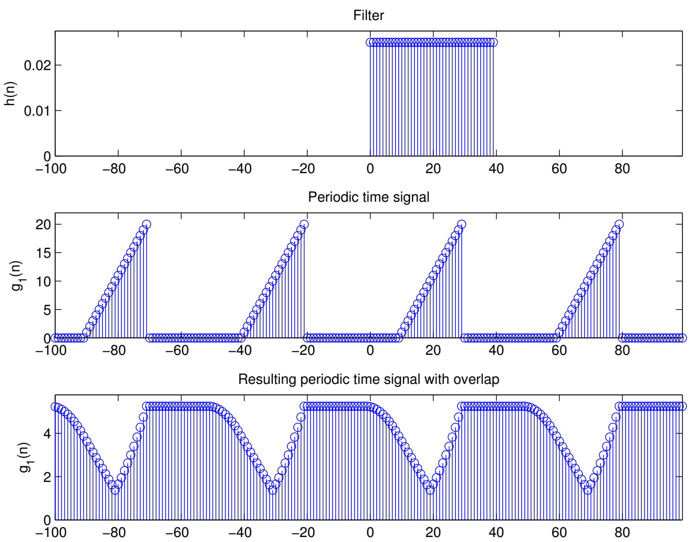
Spectrum is discrete and periodic;

$$F(u_d, v_d) = F(M - u_d, N - v_d)$$

Inverse discrete Fourier transforms:

$$f(m,n) = \sum_{u_d=0}^{M-1} \sum_{v_d=0}^{N-1} F(u_d, v_d) e^{j2\pi\left(\frac{mu_d}{M}+\frac{nv_d}{N}\right)}$$

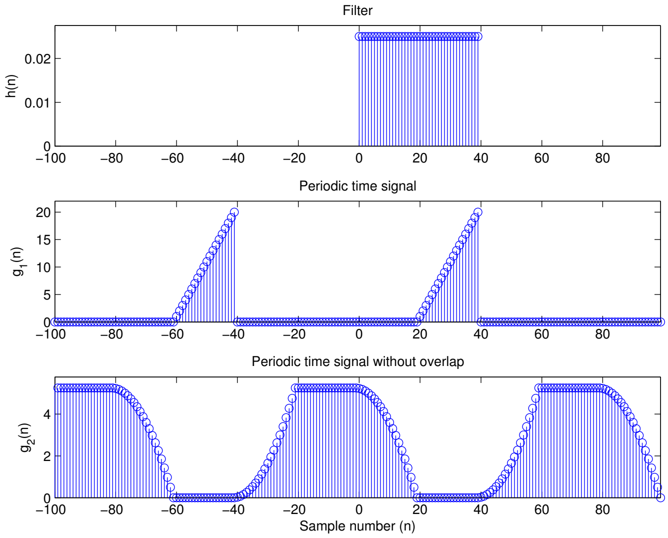
Circular convolution



N_1 - Length signal 1, N_2 - Length signal 2, N - Transform length

Here: $N < N_1 + N_2 - 1$

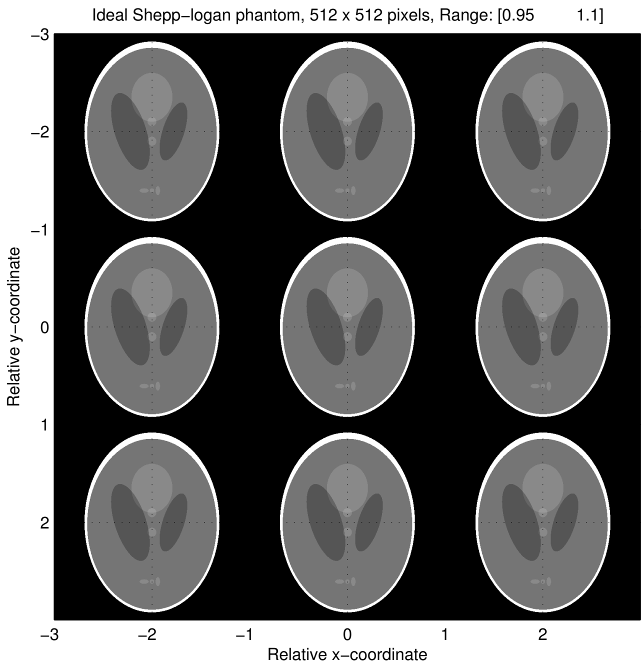
Circular convolution



N_1 - Length signal 1, N_2 - Length signal 2, N - Transform length

Here: $N > N_1 + N_2 - 1$

Circular convolution for Shepp-Logan phantom



Exercise 4: signal processing in pulsed wave system

1. Process receive signal to get complex data (load from file)

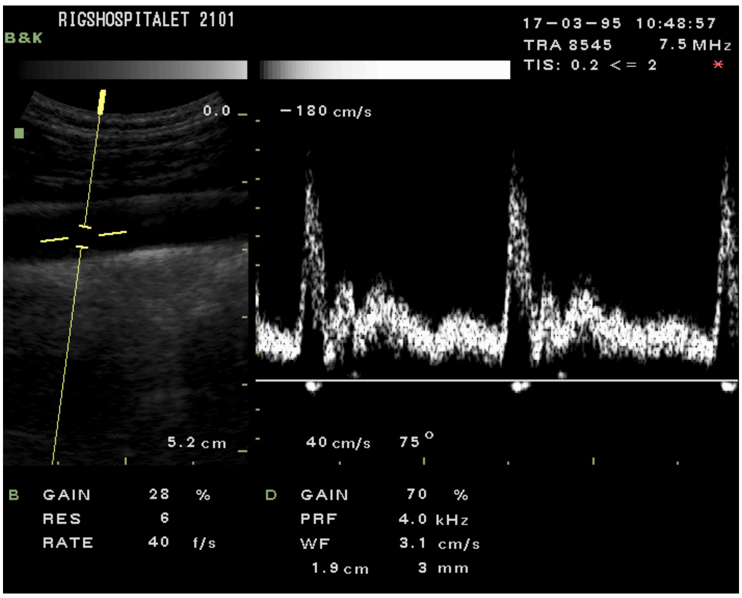
2. Divide into overlapping segements

3. Calculate power spectrum (apply compression)

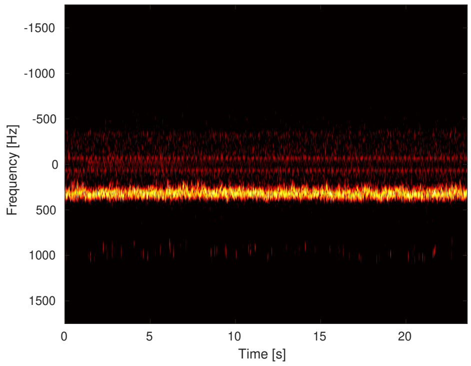
4. Display the spectra as a function of time

5. Compare the spectra for different vessels

Spectrogram from carotid artery

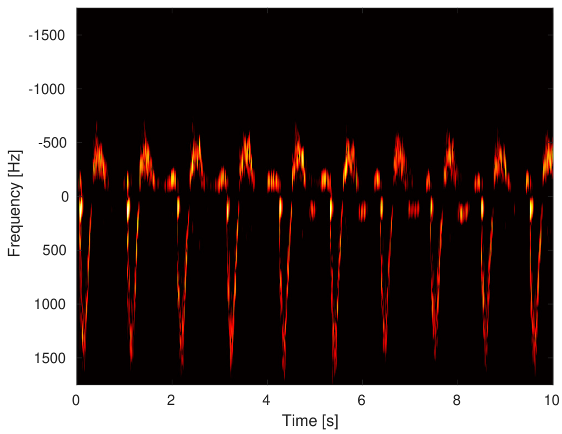


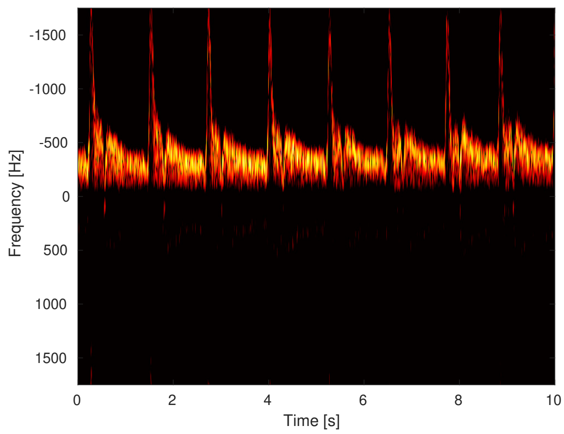
Spectrogram for phantom



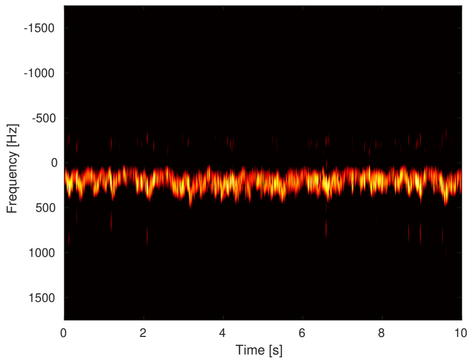
- Spectrogram for phantom data with constant parabolic flow
- 30 dB dynamic range
- 128 samples segments every 1 ms to get smooth appearance
- Hanning windowing
- FFT calculated for 1024 samples to increase smoothness

Spectrograms for aorta (left) and carotis (right)





Spectrogram for porta vein



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